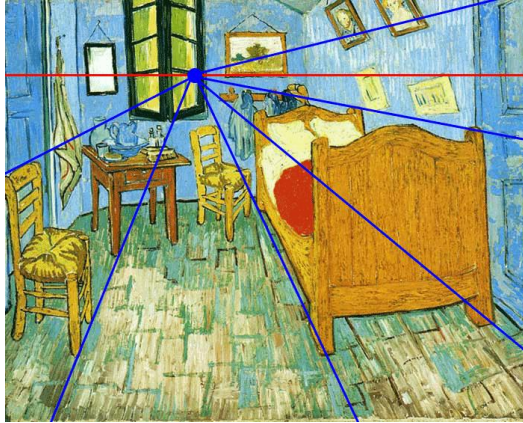


GEOMETRY



by Dr C. D. H. Cooper

Projective Geometry
Symmetry

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These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

Projective Geometry is All About Perspective

We all learnt Euclidean Geometry at school. This is the geometry that Euclid and his co-workers developed over two thousand years ago and it has to do with points, lines, circles and simple shapes like triangles and parallelograms. Euclid's methods were axiomatic. He wrote down a number of "intuitively obvious" facts and then proceeded to prove theorems in a systematic way. Over the centuries Euclidean geometry became one of the pillars of mathematics and was taught, not so much for its practical applications, but as an exercise in pure thought.

Several centuries ago René Descartes showed how the same theorems could be proved by introducing coordinates and using basic algebra. This method allowed parabolas, ellipses and hyperbolas to be treated with almost as much ease as the circle.

The theorems of Euclidean geometry mainly deal with metric properties – lengths, angles and areas. But a few theorems are independent of measurement. They deal just with incidence – points lying on lines.

Affine geometry is Euclidean geometry stripped of all measurement. Not much remains, but we can still talk of collinearity (points lying on the same line) and concurrence (lines passing through the same point) and even the concept of lines being parallel.

Parallel lines, in the affine plane, are defined to be lines that don't intersect. In Euclidean geometry they can

be viewed as lines that “remain the same distance apart” and in coordinate geometry we’d view them as lines “having the same slope”. But in affine geometry we must rely on the non-intersecting property.

Now it’s really rather bothersome that there are lines that don’t cut. Through any two distinct points there passes exactly one line. It would be nice to be able to say that any two distinct lines intersect in exactly one point, but we can’t, in the affine plane. Somehow it seems like an artificial limitation, just like being told that negative numbers don’t have square roots. Well, they don’t – not in the real number field. But what did mathematicians do there? They said “bother that – well, if they don’t exist we’ll invent them” and they proceeded to extend the real number field to the field of complex numbers.

“It’s a pity that points of intersection don’t exist for parallel lines. Well, we’ll just invent them!” And that’s exactly what happened. The affine plane was extended to the projective plane. Plane geometry was now complete in the same way that the complex numbers completed arithmetic.

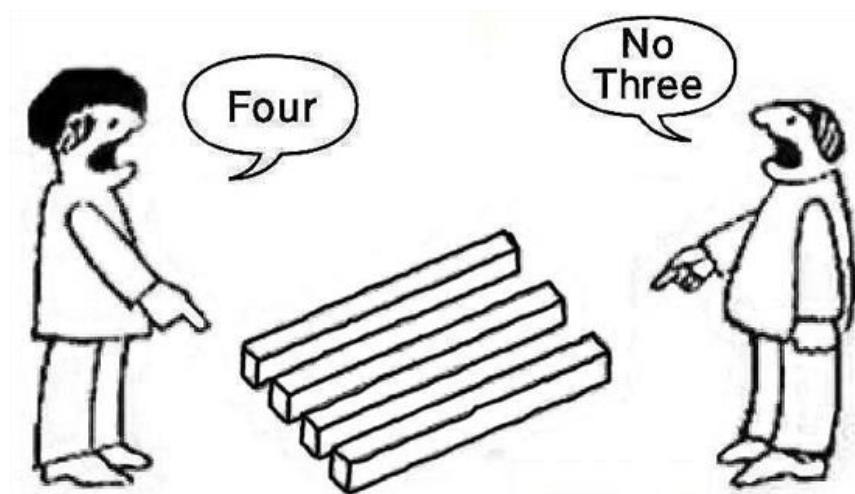
It happened in the 17th century, when perspective was being discovered by the Renaissance artists. Remember that in a perspective painting parallel lines do meet – at the horizon. Projective geometry enables incidence properties of the Euclidean plane to be proved much more easily.

As with Euclidean geometry there are two main approaches to projective geometry – through axioms and

using algebra. We've chosen to take the algebraic approach because it makes good use of linear algebra. After such an introduction to projective geometry a student truly understands linear independence and orthogonal complements.

Our proofs of Desargues' Theorem and Pascal's Theorem are built on the Collinearity Lemma, something that appears here for the first time. Moreover it turns out that a certain scalar that appears in this lemma is projective cross ratio and so the Collinearity Lemma unifies all this material. We discuss perspectivities and projectivities and the Fundamental Theorem of Projective Geometry.

ONE ASPECT OF PERSPECTIVE



Who Needs Symmetry?

Symmetry goes far beyond the mirror symmetry that everyone looks for in a beautiful face. There's rotational symmetry, translational symmetry and glide symmetry. In three dimensions there are even more types of symmetry.

Who needs symmetry? Well an expectation of symmetry seems to be hard-wired into our brains and affects the way we look at the world. In art, symmetry plays an important role in creating an expectation. It's up to the artist whether to satisfy that expectation, or to challenge it by creating beautiful asymmetry. For crystallographers, a branch of chemistry, symmetry in 3 dimensions is very much their bread and butter. Particle physicists use symmetry in higher dimensions. And architects and engineers, when using their CAD software, need to know about symmetry.

We'll explore isometries (distance preserving functions) in 2 and 3 dimensions and we'll define symmetry in terms of shapes that remain invariant under certain isometries. Groups are introduced as a tool for studying symmetry, and we describe the classification of frieze patterns (linear patterns such as you might find on a border) into seven types and wallpaper patterns (patterns that extend indefinitely in two dimensions) into 17 types.

We then look at finite rotation groups in 3 dimensions, which leads us to the seven Platonic solids.

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